MATH2060 TUTO7

Midterm Review

Q1 Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$

Show that $f'(0)$ exists and $f'(x)$ does not exist for any $x \neq 0$.

Ans: Note $\dfrac{f(t)-f(0)}{t-0} = \begin{cases} t & t \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \left| \frac{f(t)-f(0)}{t-0} \right| \leq |t| \quad \forall t \in \mathbb{R}$$

So, $f'(0) = \lim_{x \to 0} \dfrac{f(t)-f(0)}{t-0} = 0$ (by Squeeze Thm)

- Suppose $x \in \mathbb{Q} \setminus \{0\}$
  By density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists$ a seq of irrational numbers (t_n) s.t. $\lim (t_n) = x$.
  Then $\lim_{n \to \infty} \dfrac{f(t_n)-f(x)}{t_n - x} = \lim_{n \to \infty} \dfrac{0 - x^2}{t_n - x}$ does not exist
  Hence $f'(x)$ does not exist (by sequential criterion).

- Suppose $x \in \mathbb{R} \setminus \mathbb{Q}$.
  By density of $\mathbb{Q}$, $\exists$ a seq of rational numbers (s_n) s.t. $\lim (s_n) = x$.
  Then $\lim_{n \to \infty} \dfrac{f(s_n)-f(x)}{s_n - x} = \lim_{n \to \infty} \dfrac{s_n^2 - 0}{s_n - x}$ does not exist
  Hence $f'(x)$ does not exist (by sequential criterion).

Q2 Let f be a differentiable function on (a, b). Show the followings:

(i) If f is unbounded, then so is f'. Does the converse hold?

(ii) If f' is bounded, then f^2 is uniformly continuous on (a, b).

Ans: i) Suppose that f' is bounded on (a,b), say $|f'| \le M$ on (a,b)

Fix $c \in (a,b)$

By MVT, we have $\forall x \in (a,b)$,

$$f(x) - f(c) = f'(\xi)(x-c) \qquad \exists\, \xi \text{ between } c, x$$

$$\Rightarrow \quad |f(x)| \le |f(c)| + |f'(\xi)|\,|x-c|$$

$$\le |f(c)| + M(b-a) =: M',$$

contradicting the assumption that f is unbounded.

The converse is NOT true.

For example, $f(x) := \sin\left(\frac{1}{x}\right)$ for $x \in (0,1)$

is bounded by 1, but $f'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$ is unbounded

ii) By i) f' bounded $\Rightarrow$ f bounded.

Let $M_1, M_2 > 0$ s.t. $|f(x)| \le M_1$, $|f'(x)| \le M_2$ $\forall x \in (a,b)$

Now, $\forall x, y \in (a,b)$,

$$|f^2(x) - f^2(y)| = |f(x)+f(y)|\,|f(x)-f(y)|$$

$$\le 2M_1 \cdot |f(x) - f(y)|$$

$$= 2M_1 \cdot |f'(\xi)\cdot(x-y)| \qquad \exists\, \xi \text{ between } x, y \text{ by MVT}$$

$$\le 2M_1 M_2 |x-y|.$$

So f^2 is Lipschitz and hence uniformly cts on (a,b) //

Q3 Let f be infinitely differentiable function. Suppose that there is a polynomial p of degree n such that for some $\delta, C > 0$,

$$|f(x) - p(x)| \le C|x - x_0|^{n+1}, \forall x \in [x_0 - \delta, x_0 + \delta].$$

Show that p must be the n-th Taylor polynomial of f at x_0.

Ans: Claim: if a polynomial q satisfies

$$|q(x)| = |b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots + b_n(x-x_0)^n| \le C|x-x_0|^{n+1}$$

then $q \equiv 0$.

Reason: Set $x = x_0 \Rightarrow b_0 = 0$

Divide both sides by $|x-x_0|$, then let $x \to x_0$, we get $b_1 = 0$

Keep doing so, we get $b_j = 0 \quad \forall j$. //

Now, by Taylor's Thm, $\quad f(x) = P_n(x) + R_n(x)$

where $P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

and the remainder $R_n(x)$ satisfies

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1} \quad \text{for some } c \text{ between } x_0, x$$

$$\Rightarrow |R_n(x)| \le \frac{1}{(n+1)!} \underline{\sup_{x\in[x_0-\delta, x_0+\delta]} |f^{(n+1)}(x)|}\, |x-x_0|^{n+1}$$

$$=: C_1 |x-x_0|^{n+1}$$

finite since f is infinitely diff.

Let $p(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n$ be a polynomial that satisfies the assumption.

We have $\quad |P_n(x) + R_n(x) - p(x)| \le C|x-x_0|^{n+1}$

$$\Rightarrow \quad |P_n(x) - p(x)| - |R_n(x)| \le C|x-x_0|^{n+1}$$

$$\Rightarrow \left|(f(x_0) - a_0) + (f'(x_0) - a_1)(x-x_0) + \cdots + \left(\frac{f^{(n)}(x_0)}{n!} - a_n\right)(x-x_0)^n\right| \le (C_1 + C)|x-x_0|^{n+1}$$

$$\forall x \in [x_0-\delta, x_0+\delta]$$

By Claim, $\quad a_j = \frac{f^{(j)}(x_0)}{j!}, \quad j = 0, 1, \ldots, n$ //

Q4 a) Using Riemann sum and suitable partition of the interval [1,2], show that

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 .$$

b) Evaluate the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$$

Ans: a) Let $f(x) = \frac{1}{x}$.

Then f is cts on $[1,2]$. Hence $f \in \mathcal{R}[1,2]$

Consider the partition interval $[a,b]$ with tag pt. c.

Want: $f(c)(b-a) = \ln b - \ln a$

$$\frac{1}{c} = \frac{\ln b - \ln a}{b-a}$$

Such $c \in (a,b)$ exists by MVT.

Let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be an arbitrary partition of $[1,2]$.

By MVT, $\exists\, t_i \in (x_{i-1}, x_i)$ s.t.

$$\frac{1}{t_i} = (\ln x)'\big|_{x=t_i} = \frac{\ln(x_i) - \ln(x_{i-1})}{x_i - x_{i-1}}$$

Now, the tagged partition $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ satisfies

$$S(f;\dot{P}) = \sum_{i=1}^n \frac{1}{t_i}(x_i - x_{i-1}) = \sum_{i=1}^n (\ln x_i - \ln x_{i-1})$$

$$= \ln x_n - \ln x_0 = \ln 2 - \ln 1$$

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[1,2]$, $\exists\, \delta_\varepsilon > 0$ s.t. if $\|\dot{Q}\| < \delta_\varepsilon$,

then $\left| S(f;\dot{Q}) - \int_1^2 f \right| < \varepsilon$

Thus, using the tags above, we have

$$\left| \ln 2 - \ln 1 - \int_1^2 f \right| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\int_1^2 f = \ln 2 - \ln 1$ //

Q4 a) Using Riemann sum and suitable partition of the interval [1,2], show that

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 .$$

b) Evaluate the following limits:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$$

Ans: b) Observe

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} = \frac{1}{n}\sum_{j=1}^{n}\frac{n}{n+j} = \underbrace{\sum_{j=1}^{n}\frac{1}{1+j/n}\cdot\frac{1}{n}}_{\text{Riemann sum}}$$

Let $\dot{P}_n = \left\{\left[1+\frac{i-1}{n}, 1+\frac{i}{n}\right], 1+\frac{i}{n}\right\}_{i=1}^{n}$ be a tagged partition of $[1,2]$

Then $\|\dot{P}_n\| = \frac{1}{n}$

and $S(f;\dot{P}_n) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}$

Let $\varepsilon > 0$.

Choose $\delta_\varepsilon > 0$ as in a).

Take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta_\varepsilon$.

Now, $\forall n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \delta_\varepsilon$ and hence

$$\left| S(f;\dot{P}_n) - \int_1^2 f \right| < \varepsilon$$

$$\Rightarrow \quad \left| \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n} - \ln 2 \right| < \varepsilon$$

Therefore

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right) = \ln 2$$

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